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Second-quantization picture of the edge currents in the fractional quantum Hall effect

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Abstract. We study the quantum theory of two-dimensional electrons in a magnetic field and an electric field generated by a homogeneous background. The dynamics separates into a microscopic and a macroscopic mode. The latter is a circular Hall current which is described by a chiral quantum field theory. It is shown how in this second-quantized picture a Laughlin-type wave function emerges.

1 Introduction

The discovery of the fractional quantum Hall effect (FQHE) [1,2] marked a new era in condensed matter physics, both theoretical and experimental. This effect takes place in two-dimensional electron systems in a strong magnetic field. It occurs because the (Coulomb) electron–electron interaction results in the formation of highly correlated incompressible states [3], despite the fact that the lowest Landau level is only partially filled. The electron systems which demonstrate a FQHE (and are called FQH liquids) in fact represent a whole new state of matter. For its description one has to completely abandon the theories based on the single-body picture (such as the Fermi liquid theory) but use an intrinsic many-body theory, e.g. the one proposed by Laughlin [2], and develop adequate new techniques and concepts (like the one of *topological order* [4]).

All bulk excitations of the FQH liquids have a finite positive energy gap. With the gauge arguments in [5–7] one gets convinced that the FQH states should also support gapless edge excitation, similarly to the IQH case, but which, contrary to the latter, cannot be described by a chiral 1D Fermi liquid theory. In the case at hand, the topological nature of the large-scale physics of the FQH liquids provides an effective-theory description of their bulk properties by means of a topological Chern–Simons theory [8, 9,7]. In particular, based on the connection between the three-dimensional topological Chern–Simons theory and the two-dimensional chiral Wess–Zumino–Witten model (Kac–Moody algebra), discovered by Witten [10] and constructively developed in [11], it has been realized [12,8] that the edge currents of an arbitrary QH fluid in an incompressible state generate a chiral current (Kac–Moody) algebra. This observation suggested the application of methods from chiral conformal field theory to the analysis of incompressible QH liquids. In this relation one traces the nowadays popular *holographic principle* [13]. In our case it states that the topological field theory describing the scaling limit of the bulk of an incompressible QH fluid is completely determined by a chiral conformal field theory describing the edge degrees of freedom of such a fluid with the same Hall conductivity [14].

In general, the theory of the edge degrees of freedom is more complicated and less universal than the theory of the bulk [15]. Still, the edge excitations which form the so-called chiral Luttinger liquid (CLL) provide us a practical way to measure topological orders in experiment. However, most of the considerations of the edge states rest upon an effective-theory analysis (though in [16] a reformulation of the edge theory directly in terms of a set of fundamental excitations has been attempted) and we will strive for a consistent derivation from a microscopic theory.

The quantum one-dimensional anyon fields (in particular, the non-canonical fermions) constructed in [17–19], are a reasonable candidate for this role. It is our purpose in the present note to show how they originate from the initial two-dimensional Fermi algebra, what type of states they form and how the whole picture changes with the temperature. In addition, one other point will be clarified. Recall that as specially emphasized by Haldane [20], the key step in Laughlin's treatment of the FQHE has been to abandon conventional second-quantized methods, which had proved fruitless, and return to a first-quantized description. The non-canonical fermions in question in fact relate the first- and second-quantized pictures of the FQHE.

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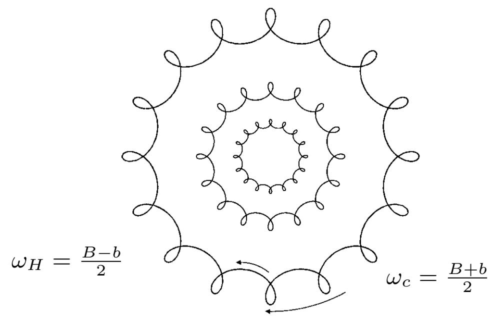

Fig. 1. Classical Larmor precession in harmonic background

More in detail, we shall consider electrons in a plane with a constant magnetic field perpendicular to it and an electric field generated by a homogeneous background harmonic potential. When only a magnetic field is present, the one-particle observables form two independent (mutually commuting) canonical pairs – the velocities and the centers of Larmor orbits. Correspondingly, the Hilbert space of the (first) quantized theory has a tensor-product structure (see also [21]). In general, an arbitrary electric field would spoil it, but it turns out that this does not happen for the particular (radial) electric field we have chosen. In this case the time evolution respects this product structure and factorizes into a microscopic and a macroscopic motion. Also upon second quantization we have a tensor product, the second factor corresponding to the edge currents mentioned above. In the thermodynamic limit one obtains a (1+1)-dimensional chiral quantum field with the possibility to use all results that are available for it. Our main goal will be to deduce all properties of the current algebra from the underlying fermionic field algebra.

2 Preliminaries

We consider the motion of electrons in two dimensions in a constant magnetic field $\boldsymbol{B}$ perpendicular to the plane of motion and an electric field $\boldsymbol{E}(\boldsymbol{x})$ generated by a homogeneous background charge. In units $e = m = 1$ the one-particle Hamiltonian is

$$H = \frac{1}{2}\left[\left(p_1 + \frac{Bx_2}{2}\right)^2 + \left(p_2 - \frac{Bx_1}{2}\right)^2\right] + \frac{E}{2}(x_1^2 + x_2^2), \tag{1}$$

(here $B, E > 0$) such that $\boldsymbol{E}(\boldsymbol{x}) = E\boldsymbol{x}$ and $B = |\boldsymbol{B}|$. Since for many particles the Coulomb repulsion cannot be treated exactly one might think that it is to some extent taken care of by a partial neutralization of the background and we consider the case $B^2 \gg E$.

The classical motion has a high-frequency mode corresponding to the cyclotron circles in B and a low-frequency rotation in the opposite direction of the centers of these circles showing the Hall effect generated by $\boldsymbol{E}$, see Fig. 1.

To separate these modes we recall that in a magnetic field, velocity components provide a pair of canonical variables

$$\boldsymbol{v} = (p_1 + Bx_2/2, p_2 - Bx_1/2) = (q, p), \ \{q, p\} = B. \tag{2}$$

Another (independent) canonical pair is given by coordinates of the centers of the cyclotron circles:

$$\bar{\boldsymbol{x}} = \left(\frac{x_1}{2} + \frac{p_2}{B}, \frac{x_2}{2} - \frac{p_1}{B}\right) = (\bar{q}, -\bar{p}), \quad \{\bar{q}, \bar{p}\} = 1/B. \tag{3}$$

In these variables the Hamiltonian (1) separates into two oscillators with an effective magnetic field $b = (B^2 + 4E)^{1/2}$ that is induced by the electric field:

$$H = [(1 + B/b)\boldsymbol{v}_b^2 + b(b - B)\bar{\boldsymbol{x}}_b^2]/4, \tag{4}$$

where in $\boldsymbol{v}_b$, $\bar{\boldsymbol{x}}_b$ (as well as in $q, p, \bar{q}, \bar{p}$ below) B is replaced by b.

For the complex coordinates

$$\begin{aligned} a &= (q + \mathrm{i}p)/\sqrt{2b}, \\ c &= (\bar{q} + \mathrm{i}\bar{p})\sqrt{b/2}, \\ a^* &= (q - \mathrm{i}p)/\sqrt{2b}, \\ c^* &= (\bar{q} - \mathrm{i}\bar{p})\sqrt{b/2}, \end{aligned} \tag{5}$$

the Poisson brackets are

$$\begin{aligned} &\{a^*, a\} = \mathrm{i}, \quad \{a, \hat{L}\} = \mathrm{i}a, \\ &\{c^*, c\} = \mathrm{i}, \quad \{c^*, \hat{L}\} = \mathrm{i}c^*, \quad \{c, a\} = \{c, a^*\} = 0, \end{aligned} \tag{6}$$

with

$$\hat{L} = x_1 p_2 - x_2 p_1 \tag{7}$$

being the generator of rotations. This makes a the high-frequency mode,

$$a(t) = \mathrm{e}^{-\mathrm{i}t(b+B)/2} a(0), \quad \hat{L}(t) = \hat{L}(0). \tag{8}$$

whereas c shows a low-frequency rotation

$$c(t) = \mathrm{e}^{-\mathrm{i}vt} c(0), \quad v = (b - B)/2. \tag{9}$$

In the limit considered $v \to E/B$, and we get the usual Hall velocity $\boldsymbol{v} = \boldsymbol{B} \times \boldsymbol{E}(x)/|\boldsymbol{B}|^2$.

In quantum theory the eigenvalues of $\boldsymbol{v}_b^2/2$ and $\bar{\boldsymbol{x}}_b^2/2$ are $b\,(n + (1/2))$, respectively $(m + (1/2))\,/b$ so that the spectrum of H becomes

$$E_{n,m} - E_0 = n(b + B)/2 + m(b - B)/2. \tag{10}$$

Upon first quantization $a, \hat{L}, c$ become operators which satisfy (6) with $\{,\} \to -\mathrm{i}[,]$. The time evolution $a(t) = \mathrm{e}^{\mathrm{i}Ht} a(0) \mathrm{e}^{-\mathrm{i}Ht}$ and similarly for $\hat{L}, c$ remains the same; see (8), (9).

For the ground state Ψ_0, $(H - E_0)\Psi_0 = 0$, we must have

$$a\Psi_0 = \hat{L}\Psi_0 = 0. \tag{11}$$

Furthermore, a^* and c decrease $\hat{L}$ by 1, so we must also have $c\Psi_0 = 0$, since by (1) $H \geq 0$. The eigenstates of H, which correspond to the eigenvalues $E_{n,m}$ from (10) are thus constructed in the usual manner:

$$\Psi_{n,m} = \frac{c^{*m} a^{*n} \Psi_0}{\sqrt{n!m!}}, \tag{12}$$

and they are simultaneously eigenstates of the angular momentum $\hat{L}$.

Let us make a few remarks.

(1) n labels the Landau levels and m shows how their degeneracy is lifted by the electric field E of the background (observe $0 < v < E/B$).

(2) $\Psi_{n,m}$ as a function of (x_1, x_2) can easily be given as in [2] and in our notations reads

$$\Psi_{n,m} = (\pi m! n! 2^{m+n+1} b^{m+n-1})^{-1/2} \mathrm{e}^{b(x_1^2+x_2^2)/4} \times \left(\frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial x_2}\right)^n \left(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2}\right)^m \times \mathrm{e}^{-b(x_1^2+x_2^2)/2}.$$

(3) The particle density in the lowest Landau level is proportional to the effective magnetic field b

$$\Psi_{0,m} = \sqrt{b/2\pi m!}(b/2)^{m/2} z^m \mathrm{e}^{-b\bar{z}z/2}, \quad \sum_m |\Psi_{0,m}|^2 = b/2\pi.$$

Thus the electric field increases the density but independent of the distance from the origin though the more distant Larmor circles are pulled further apart (Fig. 1).

3 Second quantization

In first quantization, the Hilbert space splits into a tensor product and so does the observable algebra $\mathcal{O} = \{a\} \otimes \{c\}$. The first factor corresponds to the Larmor motion and the second one to the Hall current. Upon second quantization we similarly get a tensor product of two $(1+1)$-dimensional field theories. For small E/B the first one describes the microscopic Larmor circles while the macroscopic Hall current is given by the second one, which we shall now investigate more closely.

To come to the many-body aspects we start with a Fermi field $\psi(x)$,

$$\{\psi(x), \psi^*(x')\} = \delta(x - x'), \quad \{\psi(x), \psi(x')\} = 0, \quad x \in \mathbf{R}^2 \tag{13}$$

and construct creation and annihilation operators for the various modes by $\int \mathrm{d}^2 x \Psi^*_{n,m}(x)\psi(x)$ and the hermitian conjugate. We shall start with $2M+1$ modes in the lowest Landau level $\psi_{0,m}$ and consider the limit $M \to \infty$, $E \to 0$, such that $(2M+1)v$ stays less than $(b+B)/2$ in order not to cross the next Landau level. For our result it is essential that a finite fraction of the modes are filled. Defining

$$a_{-M+m} := \int \mathrm{d}x \Psi^*_{0,m}(x)\psi(x), \tag{14}$$

we thus embed our operators in the CAR algebra $\mathcal{A}$ generated by a_m, a^*_m, $m \in \mathbf{Z}$,

$$\{a_m, a^*_n\} = \delta_{mn}, \quad \{a_m, a_n\} = 0, \tag{15}$$

and normalize the chemical potential to zero by using the ground state

$$\langle a^*_m a_n \rangle = \delta_{mn} \Theta(-m), \quad \langle a_n a^*_m \rangle = \delta_{mn} \Theta(m), \tag{16}$$

Θ being the step function (with $\Theta(0) = 1/2$). For us the relevant observable will be the Hall current as a function of the rotation angle θ. Thus we introduce the fields

$$\psi(\theta) = \sum_m a_m \mathrm{e}^{\mathrm{i}m\theta} \Theta(M - |m|) \tag{17}$$

and the current

$$j_M(\theta) = \psi^*(\theta)\psi(\theta) - \langle \psi^*(\theta)\psi(\theta) \rangle. \tag{18}$$

So far, these operators are bounded, but for $M \to \infty$ they become operator valued distributions and to get operators in this limit we have to smear them with (real) test functions; however, in the discrete case this might well be the corresponding Kronecker δ,

$$j_{p,M} = \int_{-\pi}^{\pi} \mathrm{d}\theta j_M(\theta) \mathrm{e}^{\mathrm{i}p\theta} = \sum_n : a^*_{n+p} a_n : \Theta(M - |n+p|)\Theta(M - |n|),$$

$$: a^* a : = a^* a - \langle a^* a \rangle.$$

That the limit $M \to \infty$ makes sense is shown by

Lemma 1. *If $f' \in L^2$ then $j_{p,M}$ converges for $M \to \infty$ to some j_p in the strong resolvent sense.*

The proof is as follows. Strong resolvent convergence means that $j_{p,M}$ converges strongly on a dense set of essential self-adjointness, that is, we have to show that for the corresponding vectors $|d\rangle$, $\forall \varepsilon > 0$ there exists $N \in \mathbf{Z}_+$ such that $\|(j_{p,M} - j_{p,M'})|d\rangle\| < \varepsilon \ \forall M, M' > N$. Since the strong convergence on infinitely many vectors is awkward to demonstrate we make a detour. The KMS state

$$\langle a^*_m a_n \rangle = \frac{\delta_{mn}}{1 + \mathrm{e}^{\beta m}} =: \delta_{mn} \Theta_\beta(-m) \tag{19}$$

gives in the GNS representation π_T, $T = 1/\beta$, a cyclic and separating vector $|T\rangle$ and strong convergence on $|T\rangle$ implies strong convergence on the dense set $a|T\rangle$ if $\|\tau_i(a)|\Omega\rangle\| < \infty$. In our case,

$$\|(j_M - j_{M'})a|T\rangle\|^2 = \langle T|a^*(j_M - j_{M'})^2 a|T\rangle = \langle T|\tau_i(a)a^*(j_M - j_{M'})^2|T\rangle,$$

and this goes to zero if $|(j_M - j_{M'})|T\rangle| \to 0$. Thus $\pi_T(j_M)$ converges to an operator $j_{\infty,T}$. Since matrix elements in π_T, $\langle T|\mathrm{e}^{-\mathrm{i}j_f}\mathrm{e}^{\mathrm{i}j_{\infty,T}}\mathrm{e}^{\mathrm{i}j_g}|T\rangle$ converge for $T \to 0$ and the vectors $\mathrm{e}^{\mathrm{i}j_f}|T\rangle$ are total in the Hilbert space of π_T, this defines an operator j_∞ in π_0 [17] and this is the one we shall use further on.

Now if $M > M'$,

$$j_{p,M} - j_{p,M'} = \sum_n : a^*_{n+p} a_n : \mathcal{R}^{MM'}_{n+p,n},$$

with

$$\mathcal{R}^{MM'}_{n+p,n} = \Theta(M - |n+p|)\Theta(M - |n|) - \Theta(M' - |n+p|)\Theta(M' - |n|).$$

Furthermore $\langle : a^*_m a_{m'} :: a^*_n a_{n'} : \rangle = \delta_{mn'}\delta_{nm'}\Theta_\beta(n)\Theta_\beta(-m)$ and we get

$$|\langle |j_{p,M} - j_{p,M'}|^2 \rangle| = \sum_n \left(1 + \mathrm{e}^{-\beta n}\right)^{-1} \times \left(1 + \mathrm{e}^{\beta(n+p)}\right)^{-1} \mathcal{R}^{MM'}_{n+p,n} \leq \sum_{M'-|p|}^{M-|p|} \frac{1}{1+\mathrm{e}^{-\beta n}} \cdot \frac{1}{1+\mathrm{e}^{\beta(n+p)}} \to 0.$$

□

The state (16) is nothing but the $T \to 0$ limit of the KMS state of the shift, (19). Thus the above statement carries over to (16) since everything is continuous for $T \to 0$.

Strong convergence ensures us that the limit of a product is the product of the limits. In particular, the commutator of limit elements is the limit of the commutator. Next we show that the latter is an element of the center of the strong closure of $\pi_0(\mathcal{A})$:

Lemma 2. *$\forall k$, the double commutator $[[j_{p,M}, j_{p',M}], a^*_k]$ converges to zero in operator norm.*

Let us prove this.

We use $[a^*_m a_{m'}, a^*_n a_{n'}] = a^*_m a_{n'} \delta_{nm'} - a^*_n a_{m'} \delta_{mn'}$ to conclude

$$[j_{p,M}, j_{p',M}] = \sum_n a^*_{n+p+p'} a_n \Theta(M - |n|) \times \Theta(M - |n+p+p'|) \times \left[\Theta(M - |n+p'|) - \Theta(M - |n+p|)\right].$$

Commuting with a^*_k deletes a_n and $\sum_n$, leaving us with

$$a^*_{k+p+p'} \Theta(M - |k|)\Theta(M - |k+p+p'|) \times \left[\Theta(M - |k+p'|) - \Theta(M - |k+p|)\right].$$

Now $\|a^*_{k+p+p'}\| = 1$ and hence

$$\|[[j_{p,M}, j_{p',M}], a^*_k]\| \leq \left(\Theta(M - |k| - |p'|) - \Theta(M - |k| - |p|)\right).$$

The latter differs from zero for $|k|+|p'| < M$, $|k|+|p| > M$ or $|k| + |p'| > M$, $|k| + |p| < M$, so for fixed k, p, p' and $M \to \infty$ it goes to zero. □

Lemma (2) means that whenever $\pi_0(\mathcal{A})''$, the weak closure of $\pi_0(\mathcal{A})$, has a trivial center, $[j_p, j_{p'}]$ is a c-number and the j_p's generate a bosonic current algebra. We are in this situation, but since $[j_{p,M}, j_{p',M}]$ does not converge in norm this c-number depends on the representation. It equals the limit of $\langle [j_{f,M}, j_{g,M}] \rangle$ since $\langle\rangle$ is weakly continuous and we arrive at

Theorem 1. *The operators j_p obey*

$$[j_p, j_{p'}] = -p\delta_{p,-p'}.$$

The proof is straightforward: To calculate $\langle [j_{f,M}, j_{g,M}] \rangle$ we use (16) and the expression from the previous proof. This leads to $\sum_k \Theta(M - |k|)[\Theta(M - |k-p|) - \Theta(M - |k+p|)]$. For $p > 0$ this is $-\sum_{k=-M}^{-M+p}$ and for $p < 0$ it is $\sum_{k=-M}^{-M-p}$, thus altogether we have $-p$. □

Also the two-point function of the j_p's can easily be deduced:

$$\begin{aligned}\langle j_p j_{-p'} \rangle &= \sum_{m,n} \langle : a^*_{n+p} a_n :: a^*_m a_{m+p'} : \rangle \\ &\times \Theta(M - |n|)\Theta(M - |n+p|)\Theta(M - |m|) \\ &\times \Theta(M - |m+p'|) \\ &= \sum_n \delta_{p,p'} \Theta(-n-p)\Theta(n)\Theta(M - |n|) \\ &\times \Theta(M - |n+p|) \\ &= -p\delta_{pp'}\Theta(-p).\end{aligned}$$

We have thus arrived at the current algebra and a ground state

$$\begin{aligned}&[j_p, j_{-p'}] = -p\delta_{pp'}, \quad j^*_p = j_{-p}, \quad p \in \mathbf{Z} \\ &\langle j_p j_{-p'} \rangle = -p\delta_{pp'}\Theta(-p). \end{aligned} \tag{20}$$

From this we define a density $\dot{\rho} = j$,

$$\rho(\theta) := \mathrm{i} \sum_{p \neq 0} \mathrm{e}^{-\mathrm{i}p\theta - \varepsilon|p|/2} j_p/p = \rho^+(\theta) + \rho^-(\theta), \tag{21}$$

$$\rho^*(\theta) = \rho(\theta).$$

$\varepsilon > 0$ gives a cut-off and eventually, when it has made the various manipulations legitimate, we let $\varepsilon \to 0$. The ground state $|0\rangle$ is defined as

$$\langle 0 | c^*_p = c_p | 0 \rangle = 0, \tag{22}$$

where

$$j(\theta) = \sum_{p \geq 0} \left(\mathrm{e}^{\mathrm{i}p\theta} c_p + \mathrm{e}^{-\mathrm{i}p\theta} c^*_p\right) \mathrm{e}^{-\varepsilon p/2} \sqrt{p}/2\pi.$$

For the two-point function we thus get

$$\langle \rho(\theta)\rho(\theta') \rangle = \langle \rho^-(\theta)\rho^+(\theta') \rangle = \sum_{p>0} \mathrm{e}^{\mathrm{i}(\theta-\theta')p - p\varepsilon}/p = -\ln(1 - \mathrm{e}^{\mathrm{i}(\theta-\theta') - \varepsilon}) =: \mathcal{S}(\theta - \theta'). \tag{23}$$

Now we define collective operators by

$$\Psi_\alpha(\theta) = \mathrm{e}^{\mathrm{i}\alpha\rho(\theta)}, \quad \Psi^*_\alpha(\theta) = \mathrm{e}^{-\mathrm{i}\alpha\rho(\theta)} = \Psi_{-\alpha}(\theta). \tag{24}$$

The two-point function of the operators (24) can be calculated using (22) and Hausdorff's formula since the commutator $[\rho^+(\theta), \rho^-(\theta')]$ is a c-number. For coinciding arguments it equals $\ln\varepsilon$ and thus

$$\begin{aligned}\langle \Psi_\alpha(\theta)\Psi^*_\alpha(\theta')\rangle &= \langle e^{i\alpha(\rho^+(\theta)+\rho^-(\theta))}e^{-i\alpha(\rho^+(\theta')+\rho^-(\theta'))}\rangle \\ &= e^{\alpha^2\ln\varepsilon}\langle e^{\alpha^2[\rho^-(\theta),\rho^+(\theta')]}\rangle \\ &= \varepsilon^{\alpha^2}e^{\alpha^2\langle\rho^-(\theta)\rho^+(\theta')\rangle} \\ &= e^{-\alpha^2\mathcal{S}(0)}e^{\alpha^2\mathcal{S}(\theta-\theta')}. \end{aligned} \tag{25}$$

For the general n-point function the same calculation [19] amounts to

$$\begin{aligned}&\langle \Psi_{\alpha_1}(\theta_1)\Psi_{\alpha_2}(\theta_2)\ldots\Psi_{\alpha_n}(\theta_n)\rangle \\ &\quad= e^{-\sum_{r=1}^n\alpha_r^2\mathcal{S}(0)/2-\sum_{r<s}\alpha_r\alpha_s\mathcal{S}(\theta_r-\theta_s)} \\ &\quad= \varepsilon^{\sum_{r=1}^n\alpha_r^2/2}\prod_{r<s}\left(1-e^{i(\theta_r-\theta_s)-\varepsilon}\right)^{\alpha_r\alpha_s}. \end{aligned} \tag{26}$$

To get in the limit $E\to 0$ for the time evolution $\theta\to\theta+vt$, with v as in (9), $v<E/B$, a finite velocity, we rescale $\theta = vx$, $-\pi/v\le x\le\pi/v$. Then $(1-e^{i(\theta_r-\theta_s)-\varepsilon})\to -iv(x-x')+\varepsilon$ and rescaling Ψ_α to $\Psi_\nu(x)=\varepsilon^{-\nu/2}v^{\nu/2}e^{i\nu^{1/2}\rho(x)}$ we get, e.g., the ν anyonic $2n$-point function

$$\begin{aligned}&\langle \Psi_\nu(x_1)\ldots\Psi_\nu(x_n)\Psi^*_\nu(y_1)\ldots\Psi^*_\nu(y_n)\rangle \\ &\quad= \frac{\prod_{k<l}(x_k-x_l)^\nu\prod_{k<l}(y_k-y_l)^\nu}{(-i)^{n\nu}\prod_{k,l}[(x_k-y_l+i\varepsilon)]^\nu}. \end{aligned} \tag{27}$$

This shows that for ν odd (respectively even) the Ψ fields at different points anticommute (respectively commute); however, they are not necessarily canonical. In general, in this limit Ψ_ν and Ψ^*_ν obey anyonic commutation relations [19].

A few remarks are in order. The commutation relation of the ν anyons with the local electron charge becomes

$$[\Psi_\nu(x), \rho(x')] = \sqrt{\nu}\Psi_\nu(x)\delta(x-x').$$

Thus, in this thermodynamic limit it happens that the charge generated by $\Psi_\nu(x)$ is $\nu^{1/2}\delta(x)$. This can be understood as follows. The electron charge density is $j(x)$ and $e^{i\int f(x)j(x)dx}$ changes its expectation value by $f'(x)$. Therefore, if f tends to zero at infinity, the total change in the charge would be zero. However, for the ν-anyon $\Psi_\nu(x)$ the corresponding function is $f(x)=\nu^{1/2}\Theta(x)$ and formally it induces a charge $\nu^{1/2}\delta(x)$, the opposite charge being pushed to infinity. What happens more exactly is that for the (regularized) smearing function $f_M(x)=\nu^{1/2}$ $[\Theta(x)-\Theta(x-M)]$ the unitaries $e^{i\int f_M(x)j(x)dx}$ do not converge even weakly for $M\to\infty$ but the transformation they induce does [17]. The $\Psi_\nu(x)$ are namely the ideal elements added, which generate this local gauge transformation.

Equation (25) can be written as $(\mathrm{Det}\,(1/(\mathrm{x_k}-\mathrm{y_l})))^\nu$ and shows only for $\nu=1$ the truncation properties of a quasifree state. The corresponding wave functions are given only in this case by a Slater determinant and otherwise, as we shall show below, they are of Laughlin type of order ν.

4 Anyons and Laughlin states

Definition 1. An n-particle state is given by

$$|n\rangle = \int \Psi^*(x_1)\ldots\Psi^*(x_n)|\Omega\rangle F(x_1,\ldots,x_n)dx_1\ldots dx_n;$$

its wave function is

$$\phi(x_1,\ldots,x_n) := \langle\Omega|\Psi(x_1)\ldots\Psi(x_n)|n\rangle.$$

$|n\rangle$ is a Slater state if $F(x_1,\ldots,x_n)=\prod_i f_i(x_i)$, and ϕ is of Laughlin type of order ν, if it is of the form $\prod_{i>k}(x_i-x_k)^\nu\prod_m \Phi(x_m)$, for $0<|\Phi|<\infty$ and ν odd.

Theorem 2. *For fermions of order ν a Slater state constructed with (22) has Laughlin-type wave function of order ν for a total set of f's.*

Again, let us add some remarks.

(1) Because of the anti-commutativity of the Ψ's, the Slater determinant $\mathrm{Det}\,\mathrm{f_i}(\mathrm{x_j})$ gives the same state as F.
(2) If $|\Omega\rangle$ is the vacuum then $|n\rangle=0$ if for some f_k, $\mathrm{supp}\tilde{f}_k\subset(0,-\infty)$. However, Definition 1 can also be used for KMS states and then Theorem 2 holds with some minor modification.

To prove this, we take the f's with $\mathrm{supp}\tilde{f}_k\subset(0,\infty)$ such that $f(x)$ is analytic in the upper half-plane. For these functions, $\{f_z(x)=(x-z)^{-1}, \mathrm{Im}z<0\}$ is total. Then we get up to a normalization factor

$$\begin{aligned}\phi(x_1,\ldots,x_n) &= \prod_{i>j}(x_i-x_j)^\nu \\ &\times\int\frac{dy_1}{(y_1-z_1)}\cdots\frac{dy_n}{(y_n-z_n)}\frac{\prod_{k>l}(y_k-y_l)^\nu}{\prod_{k,l}(x_k-y_l+i\varepsilon)^\nu} \\ &= \frac{\prod_{l>j}(x_l-x_j)^\nu\prod_{k>l}(z_k-z_l)^\nu}{\prod_{k,l}(x_k-z_l+i\varepsilon)^\nu}.\end{aligned}$$

Thus we have a Laughlin-type wave function with

$$\Phi(x)=\prod_l(x-z_l+i\varepsilon)^{-\nu}$$

which has the desired properties. □

We remark the following.

(1) For $\nu = 1$, ϕ is (up to a constant factor) the Slater determinant Det $\left(\mathrm{x}_\mathrm{k}^\mathrm{j} \Phi(\mathrm{x}_\mathrm{k})\right)$, for other ν's it is the ν-th power of such a determinant.

(2) For finite temperature $T = \beta^{-1}$, $\pi(x_l - x_k)$ is replaced by $\beta \mathrm{sh}[(\pi(\mathrm{x}_\mathrm{l} - \mathrm{x}_\mathrm{k}))/\beta]$ and $\Phi(x)$ by $\beta^n \prod_{l=1}^{n} \mathrm{sh}^{-\nu}[\pi(\mathrm{x} - z_l + \mathrm{i}\varepsilon)/\beta]$. By pulling out $\prod_{l>k}(x_l - x_k)^\nu$ the rest gets a factor $\prod_{l>k} \mathrm{sh}^\nu[\pi(\mathrm{x}_\mathrm{l} - \mathrm{x}_\mathrm{k})/\beta]/(\mathrm{x}_\mathrm{l} - \mathrm{x}_\mathrm{k})$ which is finite and symmetric but no longer a pointwise product.

5 Conclusions

We have studied the typical quantum Hall setting in the spirit of canonical quantum theory. The key point in our analysis is the tensor-product structure of the theory that describes it. Upon second quantization the one-dimensional field algebra related to the fractional quantum Hall effect is then exhibited in the thermodynamic limit. For its construction from some collective modes it is essential for the last Landau level to be filled to a finite fraction. This is an anyonic algebra [17–19] which, in particular, contains (non-canonical) Fermi fields characterized by odd integer values of the statistics parameter ν. Despite of being locally anticommuting, these "fermions" are unbounded, do not satisfy CAR's and their correlators exhibit a severe temperature dependence [19,22]. However, their n-particle wave functions at zero temperature are of Laughlin type of order ν, with a simple generalization for the finite-temperature case [23]. Thus, a relation between the first- and second-quantized pictures of the FQHE is achieved.

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